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DUALITY BETWEEN PLANE TRUSSES AND GRILLAGES

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Abstract—In the paper, projective plane duality, that is, a point-to-line, line-to-point, incidence-to-incidence correspondence between plane trusses and grillages of simple connection is treated. By means of linear algebra it is proved that the rank of the equilibrium matrix of plane trusses and grillages does not change under projective transformations and polarities; consequently the number of infinitesimal inextensional mechanisms and the number of independent states of self-stress are preserved under these transformations. The results obtained are also applied to structures with unilateral constraints, and by using several examples it is shown that plane tensegrity trusses have projective dual counterparts among grillages which can be physically modelled with popsicle sticks by weaving.

1. INTRODUCTION

Many problem books and text books, e.g. Beer and Johnston (1976) or Lowe (1982), on elementary engineering mechanics and on the theory of structures, present the structure in Fig. 1a as an example of how to bridge a square-shaped hole with straight beams whose length is smaller than the side-length of the hole. In Fig. 1a, the beams form a woven arrangement such that the beams support each other. The mechanical model of this structure is a horizontal plane grillage (beam grid) subjected to vertical forces, and the internal connection forces and reactions are also vertical.

The plane truss (pin-jointed framework) in Fig. 1b demonstrates many properties similar to those of the grillage in Fig. 1a. The numbers of *joints* and *bars* in Fig. 1b are the same as the numbers of *beams* and *junctions* in Fig. 1a, respectively. (Here and elsewhere in the paper, junctions mean points of intersection of beam axes and, in some cases, points of support of beams.) Three bars meet at a joint in the truss; and three junctions lie on a beam in the grillage. The structures in Figs 1a and 1b are in a correspondence called *duality*, where correspondence is between beams and joints, and junctions and bars. At a junction as well as in a bar, a single internal force arises, and the equilibrium of a beam as well as that of a joint can be described by two equations. Structures, both in Figs 1a and 1b are

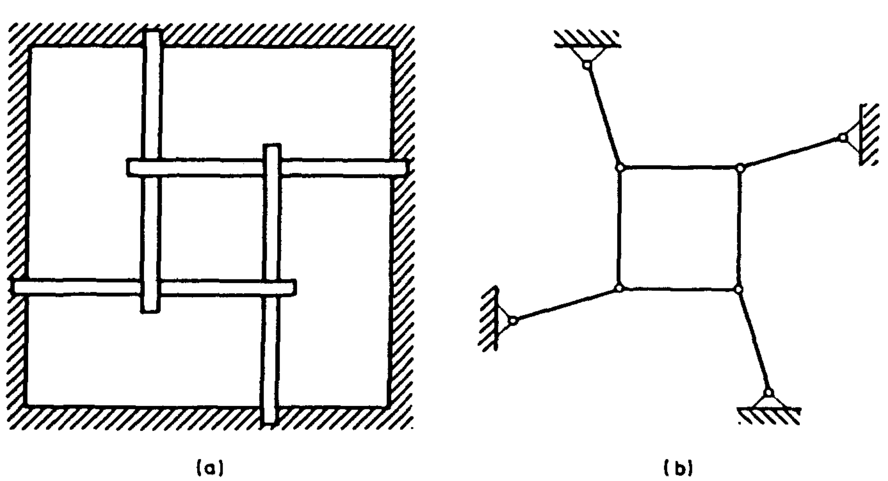

Fig. 1. (a) A grillage (beam grid). (b) A plane truss (pin-jointed framework).

statically determinate and therefore simply stiff (supposing all the junctions and members to be under bilateral constraints, i.e. able to carry both tension and compression).

It was already known, in part, in the last century that the stiffness of a structure, or more exactly, the static and kinematic properties due to the linear theory of structures (determinacy, indeterminacy, overdeterminacy) are projective properties (Rankine, 1856–1857, 1863). This means that these structural properties do not change under a projective transformation which, in our case in the plane, maps trusses into trusses and grillages into grillages.

Only recently has it been discovered in mathematics (Whiteley, 1987, 1988a, b) that these structural properties are preserved under a polarity which, in our case in the plane, maps trusses into grillages and grillages into trusses.

The most tractable plane structures are trusses, and the engineering imaging concerning trusses is well developed. Grillages, however, are less tractable and in many cases it is not apparent how a grillage works, notwithstanding—as remarked by Lowe (1982)—that the solution to a rather complicated grillage problem was already given by John Wallis in his book *Mechanica* (Wallis, 1671). So, from an engineering point of view the importance of the mathematical discovery relating to polarity is the fact that a grillage may be transformed into a truss and the static–kinematic properties of the grillage may be analyzed in an equivalent truss. But, on the other hand, in some cases polarity is able to transform trusses with very complicated networks into surprisingly simple grillages.

The aim of this paper is to show how these mathematical results can be formulated in terms of engineering mechanics and how the duality between plane trusses and grillages appears. (In order to understand the structural–mechanical consequences of projective transformations and polarities, knowledge of projective geometry is needed. But nowadays projective geometry does not belong to the mathematical tools of engineers. Therefore, in an Appendix to this paper, we define some basic terms in projective geometry.) We show projective and polar invariance of the static–kinematic properties of plane trusses and grillages. We will do it, not in a geometric, but in an algebraic way, by using vectors and matrices—which are more understandable and easier to follow by engineering readers. In fact, it will be proved that the rank of the coefficient matrix of the equilibrium equations of plane trusses and grillages does not change under projective transformations and polarities.

There is a class of bar-and-joint assemblies which are composed of members which are able to carry only tensile forces and of members which are able to carry on compressive forces. These assemblies are called "tensegrity" structures if they can be in a state of self-stress in themselves, that is, without any attachments to a foundation (Calladine, 1978). (In the mathematical literature (Roth and Whiteley, 1981) this term has a somewhat different use; and a "tensegrity" structure is composed of members with both unilateral and bilateral constraints, that is, it is formed of cables (tension members) struts (compression members) and bars (good in both tension and compression).) In this paper it will be shown that plane "tensegrity" trusses have polar counterparts among grillages where junctions are able to transmit only tensile or only compressive forces from one beam to the other. It will also be shown that physical models of these grillages can be made in the easiest way, in many cases, from "popsicle" sticks in a woven form.

In this paper "grillage" always means a grillage with simple junctions between beams, that is, with connections which are only able to carry forces perpendicular to the plane of the grillage.

2. CONCEPT OF DUALITY

It is well known that there is a duality in three-dimensional space between points and planes. So, for instance, to a polyhedron with V vertices and F faces there corresponds a dual polyhedron with V faces and F vertices such that an i-valent vertex and a j-sided face of the dual polyhedron corresponds to an i-sided face and a j-valent vertex of the primal polyhedron. Such a duality in the plane is between points and lines.

A planar configuration composed of points and lines can be considered the network of a rod structure. If points are pin joints and lines are bars then we have a plane truss. If lines are beams and points are junctions then we have a grillage. Forces and displacements are in-plane for the truss but are perpendicular to the plane for the grillage.

Figure 2a shows the network of a truss and Fig. 2b shows the network of a grillage. Joints in Fig. 2a and beams in Fig. 2b have single numbers; bars in Fig. 2a and junctions in Fig. 2b are identified by two numbers. Lines A-1, B-4, C-4 in Fig. 2a and points A-1, B-4, C-4 in Fig. 2b represent supports. Figures 2a and 2b are in duality. For example, beam 2 and junction 2-3 in Fig. 2b correspond to joint 2 and bar 2-3 in Fig. 2a, respectively. Joints 2 and 3 in Fig. 2a are joined by the bar 2-3, and beams 2 and 3 in Fig. 2b are joined at their junction 2-3. The correspondence between these structures, however, is more than simple geometric duality. Duality here gives a correspondence between certain statical and kinematical properties of the structures. For example, the truss without bar 1-3 and, dually, the grillage without junction 1-3 are statically determinate, whereas with bar 1-3 and with junction 1-3 the truss and the grillage, respectively, are statically indeterminate.

Let us investigate plane duality in a general case. *Consider first a plane truss* composed of j joints connected by b bars to each other and by a total of c kinematic constraints to a rigid foundation. If the constraints are replaced by c supporting bars then the equilibrium equations for a general joint k in a Cartesian coordinate system x, y (Fig. 3) may be written in the form

$$\sum_i (x_i - x_k)\frac{S_i}{l_i} + f_{kx} = 0,$$

$$\sum_i (y_i - y_k)\frac{S_i}{l_i} + f_{ky} = 0. \qquad (1)$$

Here x_i, y_i are the Cartesian coordinates of joint i; l_i and S_i are the length of and the force in the bar joining joints i and k; f_{kx}, f_{ky} are the components of external load at joint k. Summation has to be made for the number of bars joining at joint k. Equations (1) may be written for $k = 1, 2, \ldots, j$. After introducing the tension coefficient $t_i = S_i/l_i$ due to Southwell (1920) we obtain the set of equilibrium equations in the form

$$\mathbf{G}^{\mathrm{T}}\mathbf{t} + \mathbf{f} = \mathbf{0}. \qquad (2)$$

Here superscript T denotes transposition, $\mathbf{G}^{\mathrm{T}}$ is a $2j \times (b+c)$ matrix called equilibrium matrix ($\mathbf{G}$ is the geometric or compatability matrix; Szabó, 1973), $\mathbf{t}$ is the vector of tension coefficients, reactions included, $\mathbf{f}$ is the vector of external loads. Let us denote the rank of the matrix $\mathbf{G}^{\mathrm{T}}$ by $\rho(\mathbf{G}^{\mathrm{T}})$, the number of independent inextensional infinitesimal mechanisms

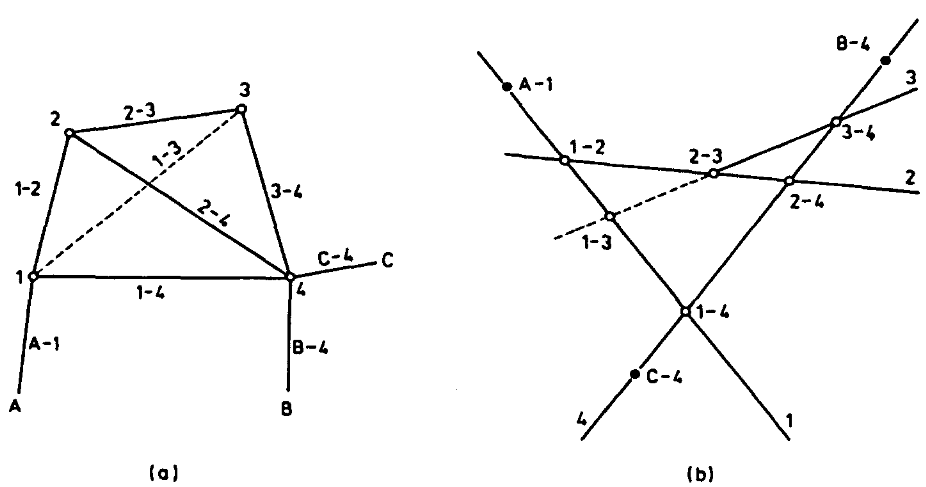

Fig. 2. Duality between the networks of (a) a plane truss and (b) a grillage.

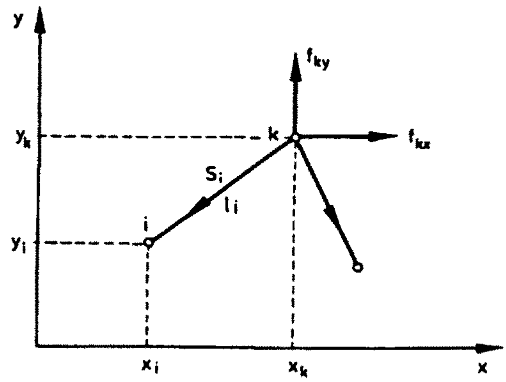

Fig. 3. A joint k which carries external forces and is connected by bars to other joints.

by m, and the number of independent states of self-stress by s; then we have (Calladine, 1978)

$$\rho(\mathbf{G}^{\mathrm{T}})+m = 2j, \quad \rho(\mathbf{G}^{\mathrm{T}})+s = b+c$$

whence

$$2j-b-c = m-s. \tag{3}$$

When we choose not to include reactions, replacing constraints, in the formulation and consider the truss to be "free in the plane" we may delete c rows and c columns of $\mathbf{G}^{\mathrm{T}}$. The equilibrium matrix of the truss in this situation will be a $(2j-c)\times b$ matrix. In many cases, only rigid body motions are intended to be prevented, so $c = 3$ and for the modified $(2j-3)\times b$ equilibrium matrix formula (3), in general, remains valid.

Consider now a grillage composed of B beams connected by J junctions to each other and by a total of C kinematic constraints to a rigid foundation. If the constraints are considered junctions between beams and a foundation then the equilibrium equations for a general beam k in a Cartesian coordinate system x, y (Fig. 4) may be written in the form

$$\begin{aligned} &\sum_i x_i' S_i' + x_f' f_k' = 0, \\ &\sum_i y_i' S_i' + y_f' f_k' = 0. \end{aligned} \tag{4}$$

Here x_i', y_i' and x_f', y_f' are the Cartesian coordinates of junction i and those of the point of application of the resultant of the external loads on beam k, respectively, S_i' is the force arising at the junction between beams i and k, and f_k' is the resultant of the external loads

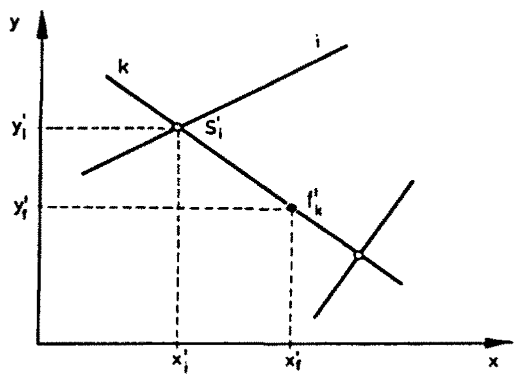

Fig. 4. A beam k which carries an external force and is connected at junctions to other beams.

on beam k. Summation has to be made for the number of junctions lying on beam k. Equations (4) are equations of moment equilibrium about the y and x axes, respectively, and it is supposed that the beam axis does not pass through the origin of the coordinate system. Equations (4) may be written for $k = 1, 2, \ldots, B$. In this way we obtain the set of equilibrium equations in the form

$$\mathbf{G}'^{\mathrm{T}}\mathbf{s}' + \mathbf{f}' = \mathbf{0}. \tag{5}$$

Here $\mathbf{G}'^{\mathrm{T}}$ is a $2B \times (J+C)$ matrix, the equilibrium matrix; $\mathbf{s}'$ is the vector of connection forces, reactions included; $\mathbf{f}'$ is the vector of external loads. Let us denote the number of independent inextensional infinitesimal mechanisms by M and the number of independent states of self-stress by S, then we have

$$\rho(\mathbf{G}'^{\mathrm{T}}) + M = 2B, \quad \rho(\mathbf{G}'^{\mathrm{T}}) + S = J + C$$

whence

$$2B - J - C = M - S. \tag{6}$$

In many cases $C = 3$ to prevent rigid body motion (three infinitesimal displacements perpendicular to the plane, which can be due to rotations about line axes in the plane), and we analyse the grillage as that which is "free perpendicularly to the plane", and for the modified $(2B-3) \times J$ equilibrium matrix eqn (6), in general, remains valid.

If there is a point-to-line and a line-to-point correspondence, that is, a plane duality in the above sense between the plane truss and grillage under investigation then

$$j = B, \quad b = J, \quad c = C$$

and from (3) and (6) it follows that

$$m - s = M - S. \tag{7}$$

Equation (7), however, does not necessarily yield the equalities $m = M$ and $s = S$. It can happen that the primal structure is statically determinate, that is, $m = 0$, $s = 0$ and so $m - s = 0$, but the dual structure, in spite of the fact that $M - S = 0$ holds, is both kinematically and statically indeterminate to degree d, that is, $M = d$, $S = d$.

This kind of plane duality, which preserves only connectivity properties, that is, the topological character of the structures, cannot be considered as a rigorous geometric plane duality in which a statement valid for points and lines remains valid by interchanging the words "point" and "line". Such a rigorous plane duality can be given by projective geometry.

There can occur hidden points of intersection of lines in the network of a structure which are not structural nodes and so are not considered. Moreover, in the dual network the points corresponding to the primal concurrent lines are not collinear, and lack of collinearity of these points is not considered either. In projective geometry such "mistakes" cannot occur since projective plane duality requires not only point-to-line and line-to-point correspondence but correspondence between all incidences.

We want to analyse the projective plane duality between plane trusses and grillages. Some basic terms and principles of projective geometry (Coxeter, 1974), which are necessary for such an investigation, are defined in an Appendix to this paper. Symbols used in the proofs are also defined there.

3. PROJECTIVE INVARIANCE OF THE RANK OF THE EQUILIBRIUM MATRIX

For plane trusses and grillages we prove the following (Rankine, 1863; Crapo and Whiteley, 1982).

Theorem. The rank of the equilibrium matrix of a structure does not change under a projective transformation.

3.1. *Trusses*

Let $\mathbf{r}_i = [x_i\ y_i\ 1]^T$, $\mathbf{r}_k = [x_k\ y_k\ 1]^T$ be the vectors of joints i and k of a truss and $\mathbf{f}_k = [f_{kx}\ f_{ky}\ 0]^T$ be the vector of the load at joint k. Using the direction unit vector $\mathbf{e}_i$ (Fig. 5) we may write the equilibrium equation of a general joint k as follows

$$\sum_i S_i \mathbf{e}_i + \mathbf{f}_k = \sum_i S_i \frac{\mathbf{r}_i - \mathbf{r}_k}{|\mathbf{r}_i - \mathbf{r}_k|} + \mathbf{f}_k = \mathbf{0},$$

that is, by using tension coefficient $t_i = S_i/|\mathbf{r}_i - \mathbf{r}_k|$,

$$\sum_i t_i(\mathbf{r}_i - \mathbf{r}_k) + \mathbf{f}_k = \mathbf{0} \tag{8}$$

which is the vector form of eqn (1) so that the third scalar equation is identically zero.

We apply a projective transformation to the truss such that $\mathbf{r}_i' = (1/\sigma_i)\mathbf{P}\mathbf{r}_i$ and we investigate how the equilibrium equations are transformed. The transformation cannot be applied directly to eqn (8) since the force vectors $t_i(\mathbf{r}_i - \mathbf{r}_k)$ and $\mathbf{f}_k$ geometrically are points at infinity but the transformation, in general, makes them finite points. We can reduce this difficulty by using the equation of the moments of the forces with respect to the point O (Fig. 5) instead of (8) (Klein, 1909; Crapo and Whiteley, 1982):

$$\left[\sum_i t_i(\mathbf{r}_i - \mathbf{r}_k) + \mathbf{f}_k\right] \times \mathbf{r}_k = \mathbf{0}, \tag{9}$$

that is,

$$\sum_i t_i(\mathbf{r}_i \times \mathbf{r}_k) + \mathbf{f}_k \times \mathbf{r}_k = \mathbf{0}. \tag{10}$$

Since point O is out of the plane of the forces, eqn (8) holds if and only if eqn (9) or (10) holds. Equation (10) in scalar form takes the shape

$$\sum_i t_i(y_i - y_k) + f_{ky} = 0,$$
$$-\sum_i t_i(x_i - x_k) - f_{kx} = 0,$$
$$\sum_i t_i(x_i y_k - x_k y_i) + f_{kx} y_k - f_{ky} x_k = 0. \tag{11}$$

It is easy to see that the first and second equations of (11) are identical to that obtained by resolving the forces along the positive y axis and the negative x axis, respectively, and the third equation describes the equilibrium of the moments of the forces about the x_3 axis. The third equation is linearly dependent on the first two equations.

In order to see the transformation of the external load we write the vector $\mathbf{f}_k$ in the form

$$\mathbf{f}_k = t_f(\mathbf{r}_f - \mathbf{r}_k). \tag{12}$$

Here $\mathbf{r}_f \neq \mathbf{r}_k$ is an arbitrary point on the line of action of the load $\mathbf{f}_k$ and $t_f = f_k/|\mathbf{r}_f - \mathbf{r}_k|$ where f_k is the magnitude of the force vector $\mathbf{f}_k$.

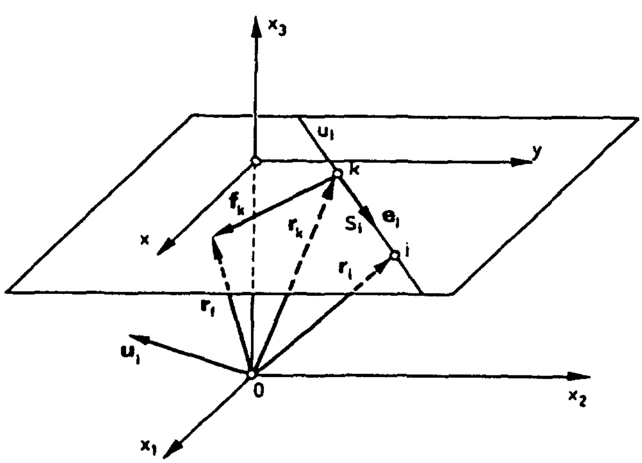

Fig. 5. Joints i and k of a truss and external and internal forces at joint k in the projective plane.

Now, consider the sum of the moments of the forces at joint k in the transformed truss with respect to point O. If (9) holds and we use the relationships (A4) and (A6), and we mark the transformed quantities by a prime, then we have

$$\begin{aligned}\left[\sum_i t_i'(\mathbf{r}_i'-\mathbf{r}_k')+\mathbf{f}_k'\right]\times\mathbf{r}_k' &= \sum_i t_i'(\mathbf{r}_i'\times\mathbf{r}_k')+t_f'(\mathbf{r}_f'\times\mathbf{r}_k')\\
&=\sum_i t_i'\alpha_i\mathbf{u}_i'+t_f'\alpha_f'\mathbf{u}_f' = \sum_i t_i'\alpha_i'\tau_i(\mathbf{P}^{\mathrm{T}})^{-1}\mathbf{u}_i+t_f'\alpha_f'\tau_f(\mathbf{P}^{\mathrm{T}})^{-1}\mathbf{u}_f\\
&=\sum_i t_i'\alpha_i'\tau_i(\mathbf{P}^{\mathrm{T}})^{-1}\frac{1}{\alpha_i}(\mathbf{r}_i\times\mathbf{r}_k)+t_f'\alpha_f'\tau_f(\mathbf{P}^{\mathrm{T}})^{-1}\frac{1}{\alpha_f}(\mathbf{r}_f\times\mathbf{r}_k)\\
&=(\mathbf{P}^{\mathrm{T}})^{-1}\left[\sum_i t_i\mathbf{r}_i\times\mathbf{r}_k+t_f\mathbf{r}_f\times\mathbf{r}_k\right]=(\mathbf{P}^{\mathrm{T}})^{-1}\left\{\left[\sum_i t_i(\mathbf{r}_i-\mathbf{r}_k)+\mathbf{f}_k\right]\times\mathbf{r}_k\right\}=\mathbf{0},\end{aligned} \tag{13}$$

that is, the transformed joint k loaded by the transformed forces is in equilibrium. Here $t_i = t_i'\alpha_i'\tau_i/\alpha_i$ and $t_f = t_f'\alpha_f'\tau_f/\alpha_f$.

Let us introduce the notation $\beta_i = \alpha_i'\tau_i/\alpha_i$. From (13) it follows that

$$\sum_i \mathbf{r}_i'\times\mathbf{r}_k' t_i' = \sum_i (\mathbf{P}^{\mathrm{T}})^{-1}(\mathbf{r}_i\times\mathbf{r}_k)\beta_i t_i'.$$

If here we extend the summation to all of the bars, writing zeros for the factors of the tension coefficients of the bars not meeting at joint k we obtain a relationship between the $3\times(b+c)$ coefficient matrix of the transformed equilibrium equations of joint k and that of the original ones. But the elements of the first two rows of these matrices are interchanged. This is apparent if we compare eqns (1) and (11). We obtain the elements of the coefficient matrices in the proper order and with the proper sign if we rearrange the elements of matrix $(\mathbf{P}^{\mathrm{T}})^{-1}$. The matrix obtained by rearranging $(\mathbf{P}^{\mathrm{T}})^{-1}$ is denoted by $\mathbf{R}$,

$$\mathbf{R} = \frac{1}{\det\mathbf{P}}\begin{bmatrix} P_{22} & -P_{21} & -P_{21}\\ -P_{12} & P_{11} & P_{11}\\ -P_{12} & P_{11} & P_{11}\end{bmatrix}$$

where P_{lm} is the cofactor of the element p_{lm} in the determinant of $\mathbf{P}$.

Let us denote the $3\times(b+c)$ extended equilibrium matrix of joint k of the transformed truss and that of the original truss by $\mathbf{H}_k'$ and $\mathbf{H}_k$, respectively, such that

$$\mathbf{H}_k' = \begin{bmatrix} \ldots & x_i'-x_k' & \ldots\\ \ldots & y_i'-y_k' & \ldots\\ \ldots & x_i'y_k'-x_k'y_i' & \ldots\end{bmatrix},\quad \mathbf{H}_k = \begin{bmatrix} \ldots & x_i-x_k & \ldots\\ \ldots & y_i-y_k & \ldots\\ \ldots & x_iy_k-x_ky_i & \ldots\end{bmatrix}. \tag{14}$$

The term "extended" refers to the fact that not two but three equilibrium equations are taken into consideration. Furthermore, let us introduce the $(b+c)\times(b+c)$ diagonal matrix $\mathbf{D}$ of coefficients β_i

$$\mathbf{D} = \langle \ldots \ \ \beta_i \ \ \ldots\rangle.$$

Then we have

$$\mathbf{H}_k' = \mathbf{R}\mathbf{H}_k\mathbf{D}. \tag{15}$$

Applying the transformation (15) for $k = 1, 2, \ldots, j$ we obtain the relationship

$$\mathbf{G}_{\mathrm{ext}}'^{\mathrm{T}} = \mathbf{L}\mathbf{G}_{\mathrm{ext}}^{\mathrm{T}}\mathbf{D}$$

where

$$\mathbf{G}_{\mathrm{ext}}'^{\mathrm{T}} = \begin{bmatrix}\mathbf{H}_1'\\ \vdots\\ \mathbf{H}_k'\\ \vdots\\ \mathbf{H}_j'\end{bmatrix},\quad \mathbf{G}_{\mathrm{ext}}^{\mathrm{T}} = \begin{bmatrix}\mathbf{H}_1\\ \vdots\\ \mathbf{H}_k\\ \vdots\\ \mathbf{H}_j\end{bmatrix},\quad \mathbf{L} = \overset{\begin{matrix}1 & 2 & \ldots & j\end{matrix}}{\begin{bmatrix}\mathbf{R} & & & \\ & \mathbf{R} & & \\ & & \ddots & \\ & & & \mathbf{R}\end{bmatrix}}. \tag{16}$$

Here $\mathbf{G}_{\mathrm{ext}}'^{\mathrm{T}}$ and $\mathbf{G}_{\mathrm{ext}}^{\mathrm{T}}$ denote the $3j\times(b+c)$ extended equilibrium matrix of the transformed truss and that of the original truss. Since $\mathbf{L}$ and $\mathbf{D}$ are nonsingular, $\rho(\mathbf{G}_{\mathrm{ext}}'^{\mathrm{T}}) = \rho(\mathbf{G}_{\mathrm{ext}}^{\mathrm{T}})$ due to Sylvester's theorem on rank (Halmos, 1974). But every third row in both $\mathbf{G}_{\mathrm{ext}}'^{\mathrm{T}}$ and $\mathbf{G}_{\mathrm{ext}}^{\mathrm{T}}$ is linearly dependent on the previous two rows, so the rank of these matrices does not change if we erase every third row. In this way we obtain the $2j\times(b+c)$ equilibrium matrices $\mathbf{G}'^{\mathrm{T}}$ and $\mathbf{G}^{\mathrm{T}}$ of the transformed truss and the original truss, respectively, such that

$$\rho(\mathbf{G}'^{\mathrm{T}}) = \rho(\mathbf{G}_{\mathrm{ext}}'^{\mathrm{T}}),\quad \rho(\mathbf{G}^{\mathrm{T}}) = \rho(\mathbf{G}_{\mathrm{ext}}^{\mathrm{T}}),$$

consequently

$$\rho(\mathbf{G}'^{\mathrm{T}}) = \rho(\mathbf{G}^{\mathrm{T}}),$$

that is, the rank of the equilibrium matrix of a truss does not change under a projective transformation, nor do the number of independent infinitesimal inextensional mechanisms and the number of independent states of self-stress: $m' = m$ and $s' = s$.

Remark. In the special case where projective transformation is an affinity, the transformation for the joint k can be directly applied on eqn (8) using notation (A9) and a relationship for $\mathbf{f}_k$ similar to (12):

$$\sum_i t_i'(\bar{\mathbf{r}}_i' - \bar{\mathbf{r}}_k') + \bar{\mathbf{f}}_k' = \sum_i t_i'(\bar{\mathbf{r}}_i' - \bar{\mathbf{r}}_k') + t_f'(\bar{\mathbf{r}}_f' - \bar{\mathbf{r}}_k')$$

$$= \sum_i t_i'[\mathbf{A}\bar{\mathbf{r}}_i + \mathbf{b} - (\mathbf{A}\bar{\mathbf{r}}_k + \mathbf{b})] + t_f'[\mathbf{A}\bar{\mathbf{r}}_f + \mathbf{b} - (\mathbf{A}\bar{\mathbf{r}}_k + \mathbf{b})]$$

$$= \mathbf{A}\Big[\sum_i t_i(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_k) + t_f(\bar{\mathbf{r}}_f - \bar{\mathbf{r}}_k)\Big] = \mathbf{A}\Big[\sum_i t_i(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_k) + \bar{\mathbf{f}}_k\Big] = \mathbf{0}.$$

Here $t_i' = t_i$ and $t_f' = t_f$, that is, *the tension coefficients do not change under an affine transformation.* For the equilibrium matrices of the truss the relationship

$$\mathbf{G}'^{\mathrm{T}} = \mathbf{B}\mathbf{G}^{\mathrm{T}} \tag{17}$$

holds, where

$$\mathbf{B} = \overset{\begin{matrix}1 & 2 & \dots & j\end{matrix}}{\begin{bmatrix} \mathbf{A} & & & \\ & \mathbf{A} & & \\ & & \ddots & \\ & & & \mathbf{A} \end{bmatrix}}. \tag{18}$$

Since $\mathbf{B}$ is nonsingular we have $\rho(\mathbf{G}'^{\mathrm{T}}) = \rho(\mathbf{G}^{\mathrm{T}})$.

3.2. *Grillages*

Consider a beam k of a grillage (Fig. 6). The force arising at its junction with beam i as well as the resultant of the external loads on the beam k are parallel to the x_3 axis and they can be considered as weights of magnitude S_i and f_k, respectively. Let the vectors of the junction and that of the point of application of the force f_k be $\mathbf{r}_i = [x_i \;\; y_i \;\; 1]^{\mathrm{T}}$ and $\mathbf{r}_f = [x_f \;\; y_f \;\; 1]^{\mathrm{T}}$. We may write the equilibrium equation of a general beam k as follows:

$$\sum_i S_i\mathbf{r}_i + f_k\mathbf{r}_f = \mathbf{0} \tag{19}$$

which is the vector form of eqn (4) so that there is a third scalar equation which is a linear combination of the first two equations. Namely, eqn (19) in scalar form takes the shape

$$\sum_i S_i x_i + f_k x_f = 0,$$

$$\sum_i S_i y_i + f_k y_f = 0,$$

$$\sum_i S_i + f_k = 0. \tag{20}$$

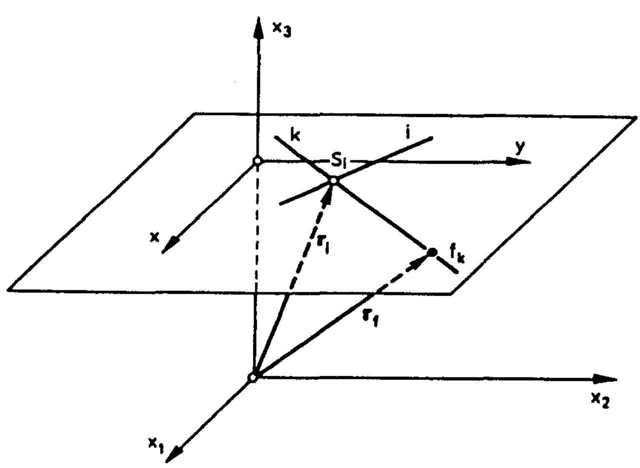

Fig. 6. Beams i and k of a grillage and external and internal forces (weights) on beam k in the projective plane.

It is easy to see that the first and second equations of (20) describe the equilibrium of the moments of the forces about the y axis and the x axis, respectively, and the third equation is the resolution of the forces along the x_3 axis.

We apply a projective transformation to the grillage, and consider the sum of the weighted vectors pointing to the beam k of the transformed grillage. If (19) holds and we use the relationship (A5), and we mark the transformed quantities by a prime then we have

$$\sum_i S_i'\mathbf{r}_i' + f_k'\mathbf{r}_f' = \sum_i S_i'\frac{1}{\sigma_i}\mathbf{P}\mathbf{r}_i + f_k'\frac{1}{\sigma_f}\mathbf{P}\mathbf{r}_f = \mathbf{P}\Big(\sum_i S_i'\frac{1}{\sigma_i}\mathbf{r}_i + f_k'\frac{1}{\sigma_f}\mathbf{r}_f\Big) = \mathbf{P}\Big(\sum_i S_i\mathbf{r}_i + f_k\mathbf{r}_k\Big) = \mathbf{0}, \tag{21}$$

that is, the transformed beam k loaded by the transformed forces is in equilibrium. Here $S_i = S_i'/\sigma_i$ and $f_k = f_k'/\sigma_f$.

From (21) it follows that

$$\sum_i \mathbf{r}_i' S_i' = \sum_i \mathbf{P}\mathbf{r}_i \frac{1}{\sigma_i} S_i'.$$

If here we extend the summation to all of the junctions, writing zeros for the coefficients of the connection forces at the junctions not lying on the beam k we obtain a relationship between the $3\times(J+C)$ extended equilibrium matrix of the transformed beam k and that of the original beam k, which has a form similar to (15):

$$\mathbf{K}_k' = \mathbf{P}\mathbf{K}_k\mathbf{E}.$$

Applying this transformation for $k = 1, 2, \ldots, B$ we have

$$\mathbf{G}_{ext}'^{\mathrm{T}} = \mathbf{M}\mathbf{G}_{ext}^{\mathrm{T}}\mathbf{E}$$

where $\mathbf{G}_{ext}'^{\mathrm{T}}$ and $\mathbf{G}_{ext}^{\mathrm{T}}$ denote the $3B\times(J+C)$ extended equilibrium matrix of the transformed grillage and that of the original grillage, such that

$$\mathbf{G}_{ext}'^{\mathrm{T}} = \begin{bmatrix}\mathbf{K}_1'\\ \vdots \\ \mathbf{K}_k' \\ \vdots \\ \mathbf{K}_B'\end{bmatrix},\quad \mathbf{G}_{ext}^{\mathrm{T}} = \begin{bmatrix}\mathbf{K}_1\\ \vdots \\ \mathbf{K}_k \\ \vdots \\ \mathbf{K}_B\end{bmatrix},\quad \mathbf{M} = \begin{array}{c} \begin{matrix}1 & 2 & \ldots & B\end{matrix} \\ \begin{bmatrix}\mathbf{P} & & & \\ & \mathbf{P} & & \\ & & \ddots & \\ & & & \mathbf{P}\end{bmatrix}\end{array}. \tag{22}$$

$\mathbf{E} = \langle \ldots \quad 1/\sigma_i \quad \ldots \rangle$ is a $(J+C)\times(J+C)$ diagonal matrix and

$$\mathbf{K}_k' = \begin{bmatrix}\ldots & x_i' & \ldots\\ \ldots & y_i' & \ldots\\ \ldots & 1 & \ldots\end{bmatrix},\quad \mathbf{K}_k = \begin{bmatrix}\ldots & x_i & \ldots\\ \ldots & y_i & \ldots\\ \ldots & 1 & \ldots\end{bmatrix} \tag{23}$$

are $3\times(J+C)$ matrices such that x_i', y_i' and x_i, y_i denote the coordinates of the junctions lying on the beam k. From this point onwards the argument is the same as that in Section 3.1 after (16). As a consequence, the rank of the equilibrium matrix of a grillage does not change under a projective transformation, nor do the number of independent infinitesimal inextensional mechanisms and the number of independent states of self-stress: $M' = M$ and $S' = S$.

Remark. In the special case where projective transformation is an affinity, the transformation for the beam k can be described by vectors in the xy plane. If we suppose that (19) holds and we use notation (A9), we have

$$\begin{aligned}\sum_i S_i'\bar{\mathbf{r}}_i' + f_k'\bar{\mathbf{r}}_f' &= \sum_i S_i'(\mathbf{A}\bar{\mathbf{r}}_i + \mathbf{b}) + f_k'(\mathbf{A}\bar{\mathbf{r}}_f + \mathbf{b})\\ &= \mathbf{A}\Big(\sum_i S_i'\bar{\mathbf{r}}_i + f_k'\bar{\mathbf{r}}_f\Big) + \mathbf{b}\Big(\sum_i S_i' + f_k'\Big)\\ &= \mathbf{A}\Big(\sum_i S_i\bar{\mathbf{r}}_i + f_k\bar{\mathbf{r}}_f\Big) + \mathbf{b}\Big(\sum_i S_i + f_k\Big) = \mathbf{0}.\end{aligned}$$

Here $S_i' = S_i$ and $f_k' = f_k$, that is, *the connection forces and the loads do not change under an affine transformation* of a grillage. The transformation of the equilibrium matrix has a form similar to (17) with the same matrix **B** in (18).

4. POLAR INVARIANCE OF THE RANK OF THE EQUILIBRIUM MATRIX

For a polarity between plane trusses and grillages we prove the following (Whiteley, 1987, 1988a)

Theorem. The rank of the equilibrium matrix of a structure does not change under a polarity.

Suppose that there is a polarity between a plane truss and a grillage so that beams i and k in Fig. 6 correspond to joints i and k in Fig. 5 and the junction of beams i and k in Fig. 6 corresponds to the bar joining joints i and k in Fig. 5, which implies that $B = j$, $J = b$, $C = c$. We use the same notation as that in Sections 3.1 and 3.2 with the difference that here the points and other quantities concerning the grillage will be marked by a prime.

Let us suppose that the joint k of the truss is in equilibrium, that is, (8) holds and we want to know whether the beam k of the grillage is in equilibrium. If we use relationships (A4), (A11) and (12) we have

$$\begin{aligned}\sum_i S_i'\mathbf{r}_i' + f_k'\mathbf{r}_k' &= \sum_i S_i'v_i\mathbf{Q}^{-1}\mathbf{u}_i + f_k'v_f\mathbf{Q}^{-1}\mathbf{u}_f \\ &= \sum_i S_i'v_i\mathbf{Q}^{-1}\frac{1}{\alpha_i}(\mathbf{r}_i\times\mathbf{r}_k) + f_k'v_f\mathbf{Q}^{-1}\frac{1}{\alpha_f}(\mathbf{r}_f\times\mathbf{r}_k) \\ &= \mathbf{Q}^{-1}\left[\sum_i t_i\mathbf{r}_i\times\mathbf{r}_k + t_f\mathbf{r}_f\times\mathbf{r}_k\right] = \mathbf{Q}^{-1}\left\{\left[\sum_i t_i(\mathbf{r}_i-\mathbf{r}_k)+\mathbf{f}_k\right]\times\mathbf{r}_k\right\} = \mathbf{0},\end{aligned} \tag{24}$$

that is, the beam k loaded by the transformed forces is in equilibrium. Here $t_i = S_i'v_i/\alpha_i$ and $t_f = f_k'v_f/\alpha_f$.

Let us introduce the notation $\gamma_i = v_i/\alpha_i$. From (24) it follows that

$$\sum_i \mathbf{r}_i'S_i' = \sum_i \mathbf{Q}^{-1}(\mathbf{r}_i\times\mathbf{r}_k)\gamma_i S_i'.$$

Extending the summation to all of the junctions and bars and writing zeros for the coefficients of the connection forces at the junctions not lying on the beam k and for the factors of the tension coefficients of the bars not meeting at joint k, respectively, we obtain a relationship between the $3\times(J+C)$ extended equilibrium matrix of the beam k and the $3\times(b+c)$ coefficient matrix of the equilibrium equations of joint k. But the elements of the first and second rows of this latter matrix are interchanged. We obtain the elements of this matrix in the proper order and with the proper sign if we rearrange the elements of matrix $\mathbf{Q}^{-1}$. The matrix obtained by rearranging $\mathbf{Q}^{-1}$ is denoted by $\mathbf{T}$,

$$\mathbf{T} = \frac{1}{\det\mathbf{Q}}\begin{bmatrix} -Q_{21} & Q_{11} & Q_{31} \\ -Q_{22} & Q_{12} & Q_{32} \\ -Q_{23} & Q_{13} & Q_{33}\end{bmatrix}$$

where Q_{lm} is the cofactor of the element q_{lm} in the determinant of $\mathbf{Q}$.

Introducing the $(b+c)\times(b+c)$ diagonal matrix $\mathbf{F}$ of coefficients γ_i

$$\mathbf{F} = \langle\ldots\ \gamma_i\ \ldots\rangle$$

and using notation in (23) and (14) we obtain the relationship between the extended equilibrium matrix of the beam k and that of the joint k in the form

$$\mathbf{K}_k' = \mathbf{T}\mathbf{H}_k\mathbf{F}. \tag{25}$$

Applying this transformation for $k = 1, 2, \ldots, B$ $(B = j)$ we have

$$\mathbf{G}'^{\mathrm{T}}_{\mathrm{ext}} = \mathbf{N}\mathbf{G}^{\mathrm{T}}_{\mathrm{ext}}\mathbf{F} \tag{26}$$

where $\mathbf{G}'^{\mathrm{T}}_{\mathrm{ext}}$ and $\mathbf{G}^{\mathrm{T}}_{\mathrm{ext}}$ are the $3B\times(J+C)$ or $3j\times(b+c)$ extended equilibrium matrices of the grillage and that of the truss. $\mathbf{G}'^{\mathrm{T}}_{\mathrm{ext}}$ is defined by (22), $\mathbf{G}^{\mathrm{T}}_{\mathrm{ext}}$ is defined by (16) and

$$\mathbf{N} = \begin{array}{c} \begin{matrix} 1 & 2 & \ldots & j \end{matrix} \\ \begin{bmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \ddots & \\ & & & \mathbf{T}\end{bmatrix}\end{array}.$$

Since $\mathbf{N}$ and $\mathbf{F}$ are nonsingular, $\rho(\mathbf{G}'^{\mathrm{T}}_{\mathrm{ext}}) = \rho(\mathbf{G}^{\mathrm{T}}_{\mathrm{ext}})$.

But every third row in both $\mathbf{G}'^{\mathrm{T}}_{\mathrm{ext}}$ and $\mathbf{G}^{\mathrm{T}}_{\mathrm{ext}}$ is linearly dependent on the two previous rows, so the rank of these matrices does not change if we erase every third row of them. In this way we obtain the $2B\times(J+C)$ or $2j\times(b+c)$ equilibrium matrices $\mathbf{G}'^{\mathrm{T}}$ and $\mathbf{G}^{\mathrm{T}}$ of the grillage and of the truss, respectively, such that

$$\rho(\mathbf{G}'^{\mathrm{T}}) = \rho(\mathbf{G}'^{\mathrm{T}}_{\mathrm{ext}}), \quad \rho(\mathbf{G}^{\mathrm{T}}) = \rho(\mathbf{G}^{\mathrm{T}}_{\mathrm{ext}})$$

and consequently

$$\rho(\mathbf{G}'^{\mathrm{T}}) = \rho(\mathbf{G}^{\mathrm{T}}),$$

that is, the rank of the equilibrium matrix of a truss and that of its polar counterpart (grillage) are the same and so are the numbers of independent infinitesimal inextensional mechanisms and the numbers of independent states of self-stress: $M' = m$, $S' = s$.

Remark. The polar invariance of the rank of the equilibrium matrix can be shown also for the case where a truss is considered the polar counterpart of a grillage, since (26) implies

$$G_{ext}^{T} = N^{-1}G_{ext}^{T}F^{-1}.$$

5. "TENSEGRITIES" AND "POPSICLE STICK GRILLAGES"

Due to the polar invariance of the rank of the equilibrium matrix, a polarity maps a plane tensegrity truss into a grillage with special junctions able to transmit only tensile forces or only compressive forces. In the case of a tensegrity truss, the members in tension and those in compression can be modelled by cables and struts, respectively. In the case of its polar counterpart, however, the junctions in tension and those in compression can be modelled by nodes of the same character, if we apply the proper connection forces on the reverse side of the beams. Figure 7 shows the possibility where all the connections in tension are replaced by connections in compression. When we make a physical model we can do this by weaving the beams in the grillage. In this way, at the junctions, only compressive forces can arise. It is very easy to demonstrate this by using popsicle sticks. (The grillage in Fig. 1(a) can also have such connections, provided the loads are acting downwards only.)

Some simple cases are presented in Fig. 8 where (a) shows woven popsicle stick grillages and (b) shows plane tensegrity trusses obtained from the grillages by polarity with respect to a circle. In (b) double lines and single lines mark members in compression and those in tension, respectively.

In Fig. 8(3a) there is a grillage which contains a junction where three beams meet. In its polar truss (Fig. 8(3b)) three joints lie on a line. The correspondence between the grillage and the truss, however, is not one-to-one. This is due to the fact that a common junction of three beams is reckoned as a *double junction*. (This fact should also be considered in the counting formula (6). In general, if n beams meet at a point, then that point is reckoned as $n-1$ junctions.) We have a one-to-one correspondence if we know which beams have direct connections in a multiple junction. In the case of the double junction in Fig. 8(3a) three kinds of connections can exist, but Fig. 8(3b) only shows the polar of one of these.

The structures in Fig. 8(5a, b) are statically three times indeterminate. The figure shows only one of the possibilities of the three independent states of self-stress.

Figures 8(6a, b) and 8(7a, b) present special cases where the popsicle stick grillages and their polar tensegrity trusses are both statically and kinematically indeterminate to degree one. Here the simultaneous indeterminancy is a consequence of the fact that in Fig. 8(6a) the six beams are tangent to a circle (in the general case, to a conic), and in Fig. 8(6b) the six joints lie on a circle (in the general case, on a conic) (Whiteley, 1988a). In Fig. 8(7a) there are two groups of beams such that the beams belonging to the same group are parallel (in the general case, are concurrent), and in Fig. 8(7b) there are two groups of joints such that the joints belonging to the same group are collinear (the joints lie on two straight lines, i.e. on a special degenerate conic). If these special properties do not hold, then the structures in Figs 8(6a, b) and 8(7a, b) become both statically and kinematically determinate (not with unilateral but bilateral constraints of the junctions and members).

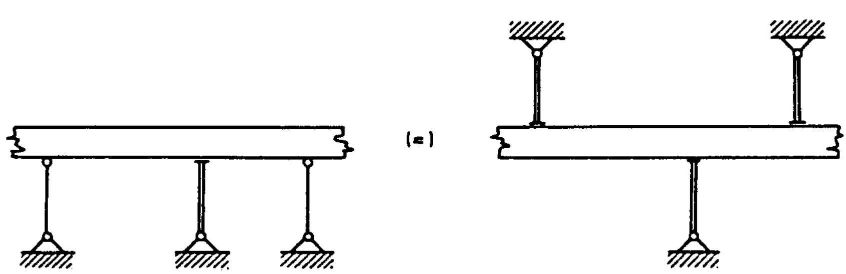

Fig. 7. Replacement of connections in tension by connections in compression.

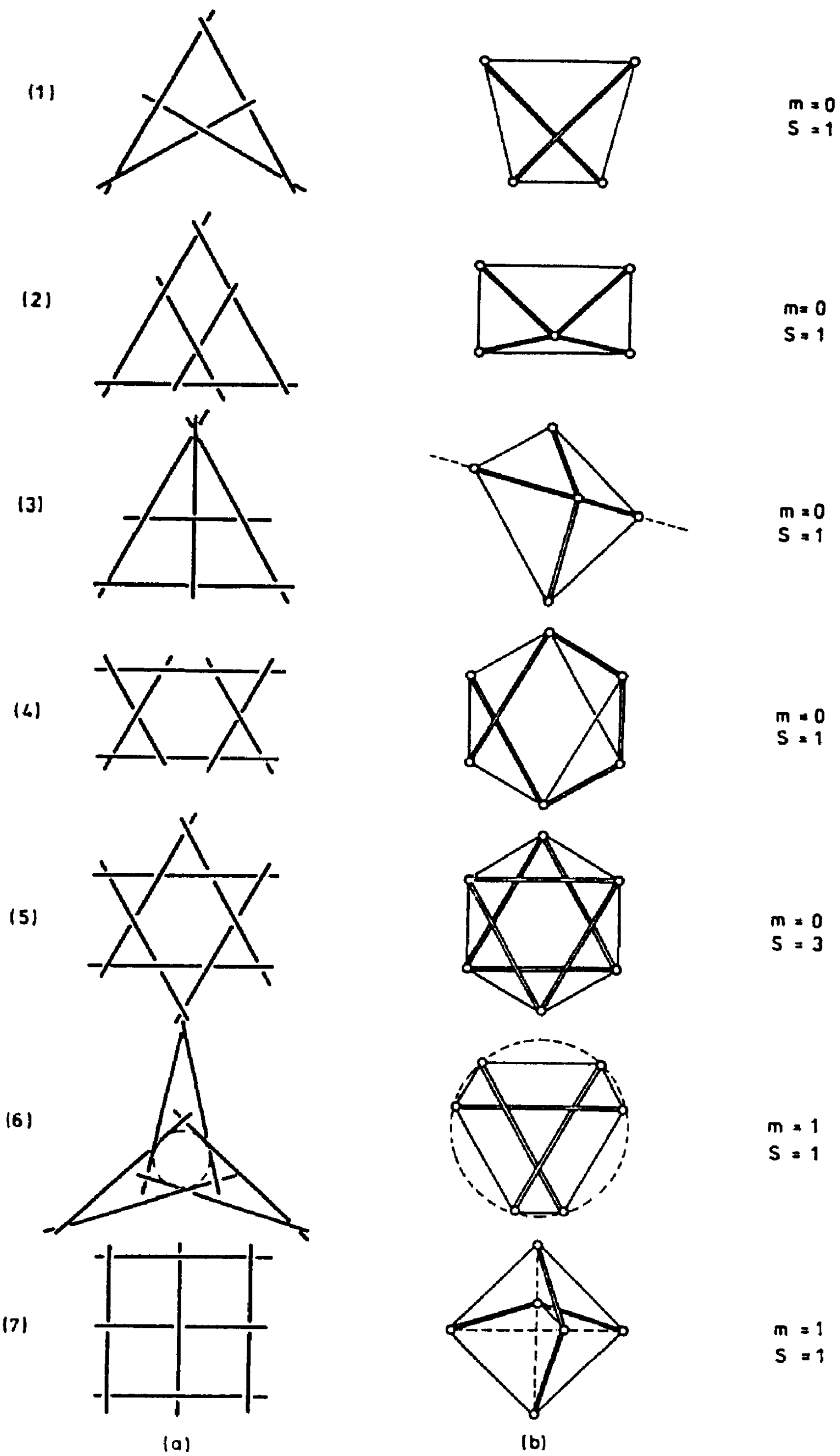

Fig. 8. Polarity with respect to a circle between (a) popsicle stick grillages and (b) plane tensegrity trusses. (1): $B = j = 4$, $J = h = 6$; (2, 3): $B = j = 5$, $J = h = 8$; (4): $B = j = 6$, $J = h = 10$; (5): $B = j = 6$, $J = h = 12$; (6, 7): $B = j = 6$, $J = h = 9$.

By using the parallel-perpendicular weaving applied in Fig. 8(7a), with little modification, one can produce an arbitrarily long popsicle stick grillage (Fig. 9) whose degree S of statical indeterminancy is one, independent of the number of sticks in the grillage, and for which $M = 0$. A polarity results in a one-fold statically indeterminate tensegrity truss whose joints corresponding to the parallel beams lie on two straight lines. Polarity with

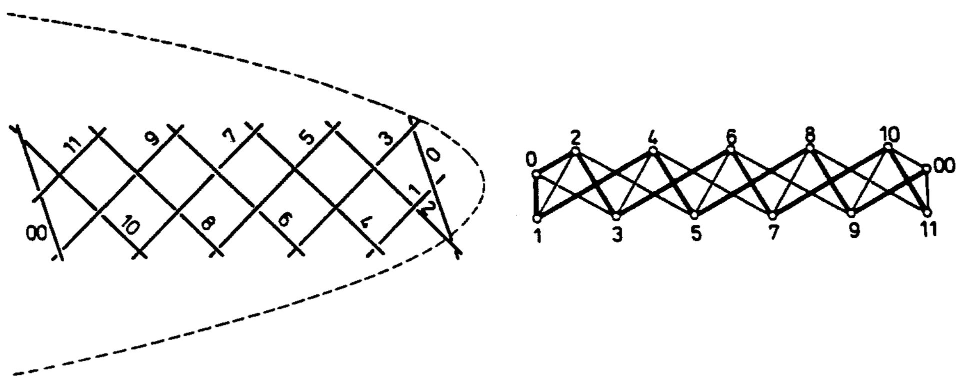

Fig. 9. Polarity with respect to a parabola between a popsicle stick grillage and a plane tensegrity truss such that $J = 2(B-1)$, $B \geq 4$ integer, or which is the same, $b = 2(j-1)$, $j \geq 4$ integer.

respect to a parabola is presented in Fig. 9 where, regardless of the ends of the structures, to a square-mesh grillage there corresponds a "regular" truss having members with inclination of ± 30 and ± 60 .

A tensegrity truss with a one-parameter state of self-stress has the property that by cutting through (by removal of) an arbitrary member of it, the state of self-stress disappears and the truss fails. A popsicle stick grillage with a one-parameter state of self-stress has the same property, but the process of disintegration is much more spectacular. This is due to the fact that in the course of weaving, the popsicle sticks are bent and, in consequence of the elastic deformation caused by bending, strain energy is stored in the grillage. When an arbitrary junction is disconnected the grillage explodes due to the stored energy. If one produces a very long "popsicle stick bomb" then one can also observe the propagation of explosion.

6. CONCLUSIONS

6.1. For plane trusses and grillages, by using only linear algebra, we have proved two invariance theorems:

(a) The rank of the equilibrium matrix of a structure does not change under a projective transformation.
(b) The rank of the equilibrium matrix of a structure does not change under a polarity.

To prove these theorems a third (redundant) equilibrium equation has been considered before doing each transformation. This is a technical trick to simplify proofs. By transforming coordinates of each node, the transformed equilibrium matrix can also be composed, and practically it does not require one to operate with three equations.

These invariance theorems imply that an infinitesimal mechanism after a projective transformation or polarity will again be an infinitesimal mechanism. Since the linear theory of structures has been applied in the treatment, it does not follow from these theorems that a finite mechanism after a projective transformation or polarity will also be a finite mechanism.

6.2. If a plane truss and a grillage are polar then they are in projective plane duality (in a point-to-line, line-to-point, incidence-to-incidence correspondence). Its reverse is, in general, not true.

6.3. If there is a polarity between a plane truss and a grillage, we can apply a projective transformation (polarity) to the plane truss and another projective transformation (polarity) to the grillage then the truss (grillage) and the grillage (truss) obtained by these transformations will be in projective plane duality.

6.4. Due to Conclusion 6.1, plane "tensegrity" trusses and "popsicle stick grillages" preserve their basic properties under a projective transformation; and plane "tensegrity" trusses are transformed into "popsicle stick grillages", and vice versa, by polarity. Consequently, there can be projective plane duality between plane "tensegrity" trusses and "popsicle stick grillages".

6.5. In this paper only the transformation of statics of plane trusses and grillages has been studied, but transformation of their kinematics, i.e. infinitesimal motions, can also be of interest. The latter is discussed in Whiteley (1988a). (The correspondence for infinitesimal motions can be given directly in linear algebra and in the geometry of parallel redrawing/perspective redrawing.)

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APPENDIX: SOME BASIC TERMS OF PROJECTIVE GEOMETRY

A *projective plane* is a three-dimensional Euclidean space in which vectors $\mathbf{r}$ and $\lambda\mathbf{r}$ are considered identical for any $\lambda \neq 0$. The vector $\mathbf{r}$ or $\lambda\mathbf{r}$ is a *point* of the projective plane. In a cartesian coordinate system x_1, x_2, x_3 a point can be given as $[x_1 \;\; x_2 \;\; x_3]^T$ or $[\lambda x_1 \;\; \lambda x_2 \;\; \lambda x_3]^T$ where the numbers x_1, x_2, x_3 or $\lambda x_1, \lambda x_2, \lambda x_3$ are ***homogeneous coordinates*** of the point.

In this paper we only deal with the real projective plane and we model it by a Euclidean plane with Cartesian coordinate system x, y such that the x and y axes are parallel to the x_1 and x_2 axes, respectively, and the origin of the coordinate system x, y is at the $x_3 = 1$ point of the x_3 axis (Fig. 10a). If a point P of the projective plane is given by the vector $[x_1 \;\; x_2 \;\; x_3]^T$ then for $\lambda = 1/x_3$ we have it, as a point of the Euclidean plane, in the form

$$\mathbf{r} = [x \quad y \quad 1]^T \tag{A1}$$

where $x = x_1/x_3$, $y = x_2/x_3$. In this paper we always use this form for denoting a point of the projective plane, except points at infinity whose vector (direction vector) has the form $[x_1 \;\; x_2 \;\; 0]^T$.

Consider the vector $\mathbf{u} = [u_1 \;\; u_2 \;\; u_3]^T$ as the normal vector of a plane passing through the point 0. This plane intersects the xy plane in a straight line u (Fig. 10b). The same line is determined by the vector $\chi\mathbf{u} = [\chi u_1 \;\; \chi u_2 \;\; \chi u_3]^T$ for any $\chi \neq 0$ real number. The vector $\mathbf{u}$ or $\chi\mathbf{u}$ is a *line* of the projective plane. For $\chi = 1/u_3$ we have

$$\mathbf{u} = [u_x \quad u_y \quad 1]^T \tag{A2}$$

where $u_x = u_1/u_3$, $u_y = u_2/u_3$. In this paper we always use this form for denoting a line of the projective plane, except the lines intersecting the x_3 axis (passing through point $[0 \;\; 0 \;\; 1]^T$), whose vectors have the form $[u_1 \;\; u_2 \;\; 0]^T$.

The point $\mathbf{r}$ and the line $\mathbf{u}$ are *incident* (that is, the point lies on the line or the line passes through the point) if their scalar product

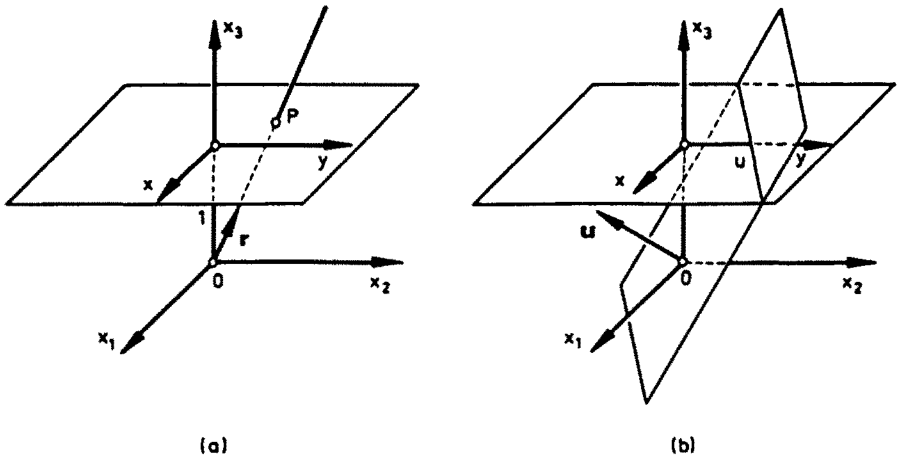

Fig. 10. (a) A point r of the projective plane. (b) A line u of the projective plane.

$$\mathbf{r}^T\mathbf{u} = 0, \tag{A3}$$

that is, $xu_x + yu_y + 1 = 0$ which is the equation of the line u in the xy plane for given u_x, u_y.

The *line joining points* $\mathbf{r}_1$ and $\mathbf{r}_2$ is the vector product

$$\alpha\mathbf{u} = \mathbf{r}_1 \times \mathbf{r}_2 \tag{A4}$$

where α is a number ($\neq 0$) such that $u_3 = 1$ is fulfilled.

A *projective transformation* is a homogeneous linear transformation of the projective plane, which transforms a point $\mathbf{r}$ into a point $\mathbf{r}'$ and a line $\mathbf{u}$ into a line $\mathbf{u}'$ in the following way:

$$\sigma\mathbf{r}' = \mathbf{P}\mathbf{r}, \tag{A5}$$

$$\tau\mathbf{u} = \mathbf{P}^T\mathbf{u}' \tag{A6}$$

where σ and τ are numbers ($\neq 0$) and $\mathbf{P}$ is the matrix of the projective transformation in the coordinate system x_1, x_2, x_3:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \tag{A7}$$

such that $\det \mathbf{P} \neq 0$.

Projective transformations *preserve incidence*. That is, if $\mathbf{r}^T\mathbf{u} = 0$ then

$$\mathbf{r}'^T\mathbf{u}' = \frac{1}{\sigma}(\mathbf{P}\mathbf{r})^T\tau(\mathbf{P}^T)^{-1}\mathbf{u} = \frac{\tau}{\sigma}\mathbf{r}^T\mathbf{P}^T(\mathbf{P}^T)^{-1}\mathbf{u} = \frac{\tau}{\sigma}\mathbf{r}^T\mathbf{u} = 0.$$

A projective transformation is called an *affinity* if in its matrix (A7) $p_{31} = p_{32} = 0$. In this case the transformation can be described in the xy plane as a linear (but not homogeneous linear) transformation:

$$\bar{\mathbf{r}}' = \mathbf{A}\bar{\mathbf{r}} + \mathbf{b} \tag{A8}$$

where

$$\bar{\mathbf{r}}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \bar{\mathbf{r}} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{A} = \frac{1}{p_{33}}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \quad \mathbf{b} = \frac{1}{p_{33}}\begin{bmatrix} p_{13} \\ p_{23} \end{bmatrix}. \tag{A9}$$

A *polarity* is a symmetric homogeneous linear transformation of the projective plane, which transforms a point $\mathbf{r}$ into a line $\mathbf{u}'$ and a line $\mathbf{u}$ into a point $\mathbf{r}'$ in the following way:

$$\eta\mathbf{u}' = \mathbf{Q}\mathbf{r}, \tag{A10}$$

$$v\mathbf{u} = \mathbf{Q}\mathbf{r}' \tag{A11}$$

where η and v are numbers ($\neq 0$) and $\mathbf{Q}$ is the matrix of polarity in the coordinate system x_1, x_2, x_3, such that $\det \mathbf{Q} \neq 0$, $\mathbf{Q}$ is symmetric.

Polarities *preserve incidence*. That is, if $\mathbf{r}^T\mathbf{u} = 0$ then

$$\mathbf{r}'^T\mathbf{u}' = v(\mathbf{Q}^{-1}\mathbf{u})^T\frac{1}{\eta}\mathbf{Q}\mathbf{r} = \frac{v}{\eta}\mathbf{u}^T(\mathbf{Q}^{-1})^T\mathbf{Q}\mathbf{r} = \frac{v}{\eta}\mathbf{u}^T\mathbf{Q}^{-1}\mathbf{Q}\mathbf{r} = \frac{v}{\eta}\mathbf{u}^T\mathbf{r} = 0.$$